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The quantum $ISL(2, \mathbb{C})$ group

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Abstract. The classical r-matrix implied by the quantum κ-Poincaré algebra of Lukierski, Nowicki and Ruegg is used to generate a Poisson structure on the $ISL(2, \mathbb{C})$ group. A quantum deformation of the $ISL(2, \mathbb{C})$ group (on the Hopf algebra level) is obtained by a trivial quantization.

Recently a quantum deformation of Poincaré algebra has been constructed which depends on dimensional parameter κ [1]. It has attractive properties like, for example, space isotropy or the existence of a natural cut-off κ. Some physical consequences of deformed Poincaré symmetry were considered by Bacry [2]. The global counterpart of this algebra, i.e. the quantum Poincaré group, was constructed by Zakrzewski [3]. However, as in the 'classical' case, for many purposes (the unitary representations of half-integer spin, supersymmetric extensions, etc) it is important to consider the universal enveloping group $ISL(2, \mathbb{C})$. It is the aim of the present paper to give a description of this group. To this end we use the approach of Zakrzewski and introduce the co-Poisson structure determined from $(1/\kappa)$-expansion of the antisymmetric part of the coproduct of the κ-Poincaré algebra. By *duality the Poisson structure on $ISL(2, \mathbb{C})$ is obtained and quantized. Due to the form* of Poisson brackets no quantization ambiguities appear. Therefore one can expect that the 'quasiclassical' (in $1/\kappa$) duality can be extended to the full quantum duality (this is the case for $E_q(2)$ (see [4])).

The classical $ISL(2, \mathbb{C})$ group consists of the pairs (a, A), where a is a fourvector and A is a matrix of the $SL(2, \mathbb{C})$ group and with the composition law

$$(a, A) * (a_1, A_1) = (a + \Lambda(A)a_1, AA_1) \tag{1}$$

where

$$\Lambda(A) = [\Lambda^{\mu}_{\nu}(A)]^{3}_{\mu,\nu=0} = [\tfrac{1}{2}\operatorname{Tr}(\sigma_\mu A \sigma_\nu A^{+})]^{3}_{\mu,\nu=0} \tag{2}$$

is the Lorentz matrix corresponding to the matrix A (and σ_μ are the Pauli matrices).

In order to determine the Poisson structure on the $ISL(2, \mathbb{C})$ group we have to calculate the right- and left-invariant vector fields. To do this we can consider first the $GL(2, \mathbb{C})$ group. Due to the fact that the r-matrix, which determines the Poisson bracket by the formula

$$\{f, g\} = 2r^{\alpha\beta}(X^{\mathrm{R}}_{\alpha} f X^{\mathrm{R}}_{\beta} g - X^{\mathrm{L}}_{\alpha} f X^{\mathrm{L}}_{\beta} g) \tag{3}$$

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where $X^{\mathrm{R}}_{\alpha}, X^{\mathrm{L}}_{\beta}$ are the right- and left-invariant vector fields, contains only the generators of $SL(2,\mathbb{C})$ we can consider all matrix elements A^{α}_{β} as independent and impose the unimodularity condition at the very end (i.e. $\det A$ has vanishing Poisson bracket with everything). In this way we obtain the following expressions for the invariant vector fields:

$$\begin{aligned}
(X^{\mathrm{L}})_{\mu} &= \Lambda^{\nu}_{\mu}(A)\frac{\partial}{\partial a^{\nu}} \\
(X^{\mathrm{L}})^{\alpha}_{\gamma} &= A^{\beta}_{\gamma}\frac{\partial}{\partial A^{\beta}_{\alpha}} \\
(\bar{X}^{\mathrm{L}})^{\alpha}_{\gamma} &= \bar{A}^{\beta}_{\gamma}\frac{\partial}{\partial \bar{A}^{\beta}_{\alpha}} \\
(X^{\mathrm{R}})_{\mu} &= \frac{\partial}{\partial a^{\mu}} \\
(X^{\mathrm{R}})^{\alpha}_{\beta} &= A^{\alpha}_{\gamma}\frac{\partial}{\partial A^{\beta}_{\gamma}} + \tfrac{1}{2}(\sigma_{\mu}\sigma_{\nu})^{\alpha}_{\beta}a^{\mu}\frac{\partial}{\partial a^{\nu}} \\
(X^{\mathrm{R}})^{\alpha}_{\beta} &= \bar{A}^{\alpha}_{\gamma}\frac{\partial}{\partial \bar{A}^{\beta}_{\gamma}} + \tfrac{1}{2}(\bar{\sigma}_{\mu}\bar{\sigma}_{\nu})^{\alpha}_{\beta}a^{\mu}\frac{\partial}{\partial a^{\nu}}
\end{aligned} \tag{4}$$

$(\mu, \nu = 0, 1, 2, 3; \alpha, \beta = 1, 2)$.

In the Lie algebra basis corresponding to the above vector fields the r-matrix obtained by Zakrzewski [3] takes the form

$$r = -\frac{\mathrm{i}}{\kappa}(L_k \otimes P_k - P_k \otimes L_k) \tag{5}$$

where

$$\begin{aligned}
L_k &= \frac{1}{2\mathrm{i}}[(\sigma_k)^{\alpha}_{\beta}X^{\beta}_{\alpha} + (\bar{\sigma}_k)^{\alpha}_{\beta}\bar{X}^{\beta}_{\alpha}] \\
P_k &= X_k
\end{aligned}$$

and $k = 1, 2, 3$.

This r-matrix induces (by duality) the following Poisson brackets of the functions on $ISL(2,\mathbb{C})$:

$$\begin{aligned}
\{f, g\} = -\frac{1}{2\kappa}\{&[(\sigma_k)^{\alpha}_{\beta}(X^{\mathrm{R}})^{\beta}_{\alpha}f + (\bar{\sigma}_k)^{\alpha}_{\beta}(\bar{X}^{\mathrm{R}})^{\beta}_{\alpha}f](X^{\mathrm{R}})_k g \\
&- [(\sigma_k)^{\alpha}_{\beta}(X^{\mathrm{R}})^{\beta}_{\alpha}g + (\bar{\sigma}_k)^{\alpha}_{\beta}(\bar{X}^{\mathrm{R}})^{\beta}_{\alpha}g](X^{\mathrm{R}})_k f \\
&- [(\sigma_k)^{\alpha}_{\beta}(X^{\mathrm{L}})^{\beta}_{\alpha}f + (\bar{\sigma}_k)^{\alpha}_{\beta}(\bar{X}^{\mathrm{L}})^{\beta}_{\alpha}f](X^{\mathrm{L}})_k g \\
&+ [(\sigma_k)^{\alpha}_{\beta}(X^{\mathrm{L}})^{\beta}_{\alpha}g + (\bar{\sigma}_k)^{\alpha}_{\beta}(\bar{X}^{\mathrm{L}})^{\beta}_{\alpha}g](X^{\mathrm{L}})_k f\}.
\end{aligned} \tag{6}$$

If we perform the standard quantizations of the Poisson brackets of the coordinate functions on $ISL(2,\mathbb{C})$, by replacing $\{\,,\} \to \frac{1}{\mathrm{i}}[\,,]$, one obtains the following set of commutation relations

$$
\begin{aligned}
&[A^\alpha_\beta, A^\gamma_\delta] = 0 \qquad [\bar{A}^\alpha_\beta, \bar{A}^\gamma_\delta] = 0 \qquad [\bar{A}^\alpha_\beta, A^\gamma_\delta] = 0 \\
&[a^k, a^j] = 0 \qquad [a^k, a^0] = \frac{\mathrm{i}}{\kappa} a^k \\
&[A^\alpha_\beta, a^0] = \frac{\mathrm{i}}{2\kappa} (A\sigma_k)^\alpha_\beta \Lambda^0_k(A) \\
&[A^\alpha_\beta, a^k] = \frac{\mathrm{i}}{2\kappa} [(A\sigma_n)^\alpha_\beta \Lambda^k_n(A) - (\sigma_k A)^\alpha_\beta] \\
&[\bar{A}^\alpha_\beta, a^0] = \frac{\mathrm{i}}{2\kappa} (\bar{A}\bar{\sigma}_k)^\alpha_\beta \Lambda^0_k(A) \\
&[\bar{A}^\alpha_\beta, a^k] = \frac{\mathrm{i}}{2\kappa} [(\bar{A}\bar{\sigma}_n)^\alpha_\beta \Lambda^k_n(A) - (\bar{\sigma}_k \bar{A})^\alpha_\beta].
\end{aligned}
\tag{7}
$$

Since the composition law is compatible with Poisson bracket, the above commutation rules are compatible with the following coproduct:

$$
\begin{aligned}
&\Delta(A^\alpha_\beta) = A^\alpha_\gamma \otimes A^\gamma_\beta \\
&\Delta(a^\mu) = \Lambda^\mu_\nu(A) \otimes a^\nu + a^\mu \otimes I.
\end{aligned}
\tag{8}
$$

The antipode is given by

$$
S((a, A)) = (-\Lambda^\mu_\nu(A^{-1})a^\nu, A^{-1}). \tag{9}
$$

We conclude that relations (7)–(9) define a Hopf algebra, the $ISL_\kappa(2, \mathbb{C})$ group. Note that the map

$$
(a, A) \to (a, \Lambda(A)) \tag{10}
$$

is the Hopf algebra homomorphism from $ISL_\kappa(2, \mathbb{C})$ to the κ-Poincaré group of Zakrzewski.

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